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Averaging of a finely laminated elastic medium with roughness or adhesion on the contact surfaces of the layers☆

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ABSTRACT

The governing relations of a laminated elastic medium with non-ideal contact conditions in the interlayer boundaries are obtained by an asymptotic averaging method. The interaction of rough surfaces is described by a non-linear contact condition which simulates the local deformation of the microroughnesses using a certain penetration of the nominal surfaces of the elastic layers. The cohesive forces, caused by the thin adhesive layer, are described within the limits of the Frémond model which includes a differential equation characterizing the change in the cohesion function. A piecewise-linear approximation of the initial positive segment of the Lennard–Jones potential curve is proposed to describe of the adhesive forces between smooth dry surfaces. A comparison is made with the solution obtained within the limits of the Maugis–Dugdale model based on a piecewise-constant approximation. Solutions of the above problems are constructed taking account of the possible opening of interlayer boundaries.

The problem of the averaging of a laminated elastic structure with linearly elastic slippage conditions has been considered earlier[1]. Problems with unilateral contact conditions and coulomb friction have been studied in the case of various periodic structures (a laminated medium and a block medium composed of periodically arranged cells of the bonded "brick masonry" or "paved surface" type.[2,3] Non-linear conditions of a generalized type (linearly elastic slippage and viscous friction or non-linear friction which, in the limit, obeys Coulomb's law) have been assumed,[4] taking account of the possible separation of the interlayer boundaries.

A reduction in the thickness of the layers in a practical laminated structure to dimensions comparable with the scale of the roughness on the surfaces of the layers leads to the need to formulate non-linear boundary conditions which simulate the contact between real rough surfaces. Such boundary conditions in problems of the contact of an elastic body with a rigid obstruction have been used in papers of a mathematical character[5,6] (see also the reviews in Refs. 7 and 8). Contact problems for a rough elastic half space have been considered earlier in many applied papers (see the review in Ref. 9).

As the roughness is reduced, the contact between the layers becomes more rigid. However, as the degree of cleanliness of contacting plane surfaces increases, the effect of adhesion begins to play an important role. Besides, the cohesive forces between the layers (which allow of some separation of the interlayer boundaries) can be created artificially by introducting a thin adhesive polymer layer. Different models (see Refs 10 and 11, for example) have been proposed to describe the adhesive forces.

A previously used model[12,13] is employed below. Its distinguishing feature is the introduction of a cohesion function characterizing the evolution of the state of the adhesive bonds. A piecewise-constant approximation of the adhesive forces (the Maugis–Dugdale model), which has been used earlier in papers on the contact of elastic bodies[14] and crack mechanics,[15] is also considered.

The asymptotic averaging method,[16,17] which leads to non-linear problems in a periodicity cell, is used to solve the problem. These problems can be explicitly solved in the case of the stratified medium considered. It has been pointed out [2] that the problem for a periodicity cell with friction conditions can be solved numerically in the case of a block medium of complex structure. The solution of the problem for a periodicity cell has been obtained[18] in the problem of the torsion of an elastic rod with friction conditions on the boundaries of the longitudinal fibres. The limiting relations for a block medium, where, when they are satisfied, it loses its load-bearing capacity, have been investigated in detail[3] in the plane case. The scheme for constructing the asymptotics follows Nikitin's approach[4] in certain details. The solution found earlier[2] is obtained on taking the limit in the special case of slight roughness without adhesion.

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Note that the transfer of the results which have been obtained to problems of the deformation of laminated plates (see Refs 19 and 20, for example) with non-ideal contact conditions in the interlayer boundaries is also of practical interest.The existence of a small parameter, equal to the ratio of the thickness of a layer to the thickness of the packet, enables one to separate out both the macroscopic as well as the microscopic structures (according to the well-known terminology[16]) of the stress-strain state, which are responsible for the corresponding levels of the texture of the laminated medium. The roughness of surfaces of the elastic layers determines a third level (the second order microlevel), characterized by a further parameter with the dimension of length. (This also appears when there are adhesive forces.) It is the situation when the parameter determining the roughness scale and the characteristic length of the adhesive bonds are comparable with the thickness of the layers that is considered below. In this case, the governing relations of a finely stratified medium at the macrolevel carry within themselves an imprint of the second order microstructure.

1. Contact between elastic layers with normal compliance of the roughness

Consider a laminated elastic medium, the layers of which are packed perpendicular to the Ox_3 axis. The layers are made of the same material which is homogeneous and possesses isotropic elastic constants and, moreover, all the layers have the same thickness h.

Suppose a packet of thickness H is charged with surface loads which act on the faces $x_3 = 0$ and $x_3 = H$. With such a choice of the position of the origin of the system of coordinates, the interfaces of the layers $\Gamma^{(s)}$ coincide with the planes $x_3 = sh$ ($s = 1, 2, \ldots$). We fix the thickness of the packet H and, assuming that the thickness of the layers h is a variable quantity, we introduce the small parameter $\varepsilon = h/H$.

We will now assume that, at an interface $\Gamma^{(s)}$, the layers interact with one another in accordance with the law of unilateral contact with friction and the normal pressure is determined taking account of the deformation of the roughness. In fact, in the case of peeling, we have

$$[u_3] \geq 0 \Rightarrow \sigma_{33} = 0 \tag{1.1}$$

where $[u_3] = u_3|_{x_3=x_3^{(s)}+0} - u_3|_{x_3=x_3^{(s)}-0}$ is the discontinuity in the normal displacement.

When the surfaces come into contact, we shall have

$$[u_3] < 0 \Rightarrow \sigma_{33} = -c_m(-[u_3])^m \tag{1.2}$$

The roughness parameters c_m and m can be expressed in terms of the characteristics of the reference curve of the profile of the rough surface. Using the Zhuravlev–Kragel'skii–Demkin theory, [21] the value $m = 2\nu - 1/2$ was obtained, where v is the exponent of the Demkin power law. For the most common values $\nu \in [2,3]$, we obtain $m \in [3/2,5,2]$. It is interesting to note (see Ref. 22, §3.3.4) that the exponent v is determined by the fractal properties of the surface.

Using the function $(t)_+ = (t + |t|)/2$, we reduce relations (1.1) and (1.2) to a single relation

$$\sigma_{33} = -c_m(-[u_3])_+^m \tag{1.3}$$

The condition for the tangential interaction of the contacting surfaces is determined by Coulomb's law

$$|\boldsymbol{\sigma}_\tau| + f\sigma_{33} \leq 0, \quad (|\boldsymbol{\sigma}_\tau| + f\sigma_{33})[\mathbf{u}_\tau] = 0, \quad \boldsymbol{\sigma}_\tau \cdot [\mathbf{u}_\tau] = |\boldsymbol{\sigma}_\tau||[\mathbf{u}_\tau]| \tag{1.4}$$

Here, $\boldsymbol{\sigma}_\tau = (\sigma_{31}, \sigma_{31}, 0)$ is the shear stress vector, $\mathbf{u}_\tau = (u_1, u_2, 0)$ is the tangential displacement vector, $[\mathbf{u}_\tau]$ is the discontinuity in the tangential displacement vector at the interface, f is the coefficient of friction and $|\boldsymbol{\sigma}_\tau| = (\sigma_{31}^2 + \sigma_{32}^2)^{1/2}$ is the modulus of the vector $\boldsymbol{\sigma}_\tau$.

On the surface $\Gamma^{(s)}$, it is possible, generally speaking, to separate out the segment $\Gamma_c^{(s)}$ of loaded contact in which the boundary condition (1.2) is satisfied, the segment $\Gamma^{(s)}$ in which there is no contact of the surfaces and the conditions $[u_3] > 0$ and $\sigma_{33} = 0$ are satisfied, and the segment $\Gamma_0^{(s)}$ of contact between the surfaces without stress transfer when the conditions $[u_3] = 0$ and $\sigma_{33} = 0$ are simultaneously satisfied. Boundary condition (1.1) determines the set $\Gamma_-^{(s)} \cup \Gamma_0^{(s)}$.

Returning now to inequality (1.4), we see that the condition $\sigma_{33} = 0$ implies the equality $|\boldsymbol{\sigma}_\tau| = 0$ and condition (1.1) can thereby be refined:

$$[u_3] \geq 0 \Rightarrow \sigma_{3i} = 0, \quad i = 1, 2, 3 \tag{1.5}$$

In turn, the segment of loaded contact, according to boundary condition (1.4), is subdivided into a zone of cohesion $\Gamma_{ca}^{(s)}$, where the conditions

$$[u_3] < 0, \quad |\boldsymbol{\sigma}_\tau| < fc_m(-[u_3])^m, \quad [\mathbf{u}_\tau] = 0 \tag{1.6}$$

are satisfied, a slippage zone $\Gamma_{cs}^{(s)}$, where, assuming that $[\mathbf{u}_\tau] \neq 0$, we shall have

$$[u_3] < 0, \quad |\boldsymbol{\sigma}_\tau| = fc_m(-[u_3])^m, \quad \frac{\boldsymbol{\sigma}_\tau}{|\boldsymbol{\sigma}_\tau|} = \frac{[\mathbf{u}_\tau]}{|[\mathbf{u}_\tau]|} \tag{1.7}$$

and, finally, a limiting equilibrium zone $\Gamma_{c0}^{(s)}$, where the conditions

$$[u_3] < 0, \quad |\boldsymbol{\sigma}_\tau| = fc_m(-[u_3])^m, \quad [\mathbf{u}_\tau] = 0 \tag{1.8}$$

are satisfied.

Note that the static equilibrium conditions

$$[\sigma_{33}] = [\sigma_{31}] = [\sigma_{32}] = 0 \tag{1.9}$$

were taken into account when writing down boundary conditions (1.3) and (1.4).

In the situation of a common position of the sets $\Gamma_0^{(s)}$ and $\Gamma_{c0}^{(s)}$, for which the extra boundary conditions are put in place, there are lines of change of the type of boundary conditions. It has to be emphasized beforehand that the asymptotic models, obtained as a result of using the averaging method, are incapable of describing the behaviour of a laminar medium in the neighbourhoods of the sets $\Gamma_0^{(s)}$ and $\Gamma_{c0}^{(s)}$.

It has been noted earlier[7] that, on taking the limit $c_m \to +\infty$ in the solution of a contact problem with the normal compliance conditions (1.3), a solution of the problem is obtained with the following unilateral contact conditions

$$[u_3] \geq 0, \quad \sigma_{33} \leq 0, \quad \sigma_{33}[u_3] = 0 \tag{1.10}$$

The second inequality of (1.10) guarantees that there are no normal tensile stresses in $\Gamma^{(s)}$ and the first inequality allows of the opening of the contacting surfaces (with the formation of cavities) and simultaneously prohibits their mutual penetration. Conditions (1.10) therefore correspond to the case of negligibly small roughness.

With respect to the stiffness c_m, we make the following assumption

$$c_m = \varepsilon^{-m} c_m^* \tag{1.11}$$

where the quantity c_m^* is independent of the parameter ε. It is clear that the value of the coefficient c_m must be large enough to prevent perceptible mutual penetrations of the contacting surfaces. The motivation for choosing explicit relation (1.11) is explained by the fact that, under certain assumptions, the coefficient c_m is inversely proportional to the maximum height of the microroughnesses to the power m.

2. Contact of elastic layers when there is adhesion

Suppose thin adhesive seams are introduced between the layers when forming the packet. We shall assume that the thickness of these seams is negligibly small compared with the thickness of the layers. Following the well known approach[23], at an interface $\Gamma^{(s)}$ we specify a cohesion function $\beta(t, x_1, x_2, x_3^{(s)})$, which depends on the time-like parameter t and the coordinates, for the purpose of characterizing the intensity of the cohesion between the contacting surfaces.

The quantity β takes values from 0 to 1 and, when $\beta = 0$, all the bonds are assumed to be broken, while a value $\beta = 1$ corresponds to an entirety of adhesive bonds. When $t = 0$, the quantity β takes a specific initial value $\beta_0 \in [0,1]$. It is assumed that the evolution of the state of the adhesive seam is described by the differential equation

$$\dot{\beta} = -\gamma_{\text{de}}(R([u_3]))^2 \beta \tag{2.1}$$

A derivative with respect to the parameter t is denoted by a dot and $0 < \gamma_{\text{de}}$ is an experimentally determined constant characterizing the rate of breaking of the bonds. The shear function $R(\upsilon)$ is defined by the formula

$$R(\upsilon) = (\upsilon)_+ - (\upsilon - L)_+ = \begin{cases} 0, & \upsilon \leq 0 \\ \upsilon, & \upsilon \in [0, L] \\ L, & \upsilon \geq L \end{cases} \tag{2.2}$$

The physical meaning of the characteristic length of the adhesive bonds, above which, when the opening $[u_3]$ is increased, the bonds of the adhesive seam make no contribution to the magnitude of the adhesive force, can be given to the constant L. Note that bond breaking is irreversible in the model being considered since, as follows from Eq. (2.1), the rate of change of the cohesion function $\dot{\beta} \leq 0$.

Henceforth, we shall assume that the relative normal displacement of the contacting surfaces of two neighbouring elastic layers occurs in accordance with the boundary conditions

$$[u_3] \geq 0, \quad \sigma_{33} - \gamma_n \beta^2 R([u_3]) \leq 0, \quad (\sigma_{33} - \gamma_n \beta^2 R([u_3]))[u_3] = 0 \tag{2.3}$$

where $0 < \gamma_n$ is an experimentally determined constant characterizing the tensile stiffness of the adhesive bonds.

It can be seen from the first inequality of (2.3) that, in the model being considered, mutual penetration of the contacting surfaces is not permitted and, moreover, that, when $[u_3] = 0$, according to the definition of the shear function (2.2), we obtain $\sigma_{33} \leq 0$. In other words, the stresses between the contacting surfaces cannot be tensile stresses. At the same time, the state of the adhesive seam does not change, that is, $\dot{\beta} = 0$. We also note that, when the cohesion function reaches a value of zero (all the bonds of the adhesive seam have been broken), the unilateral contact conditions (2.3) become the Signorini conditions (1.10).

The adhesive seam not only restrains the surfaces in the case of their normal stretching but also opposes their relative tangential displacement. We shall assume that relative tangential displacement of the surfaces of the elastic layers occurs according to the boundary condition

$$\boldsymbol{\sigma}_\tau = \gamma_\tau \beta^2 \mathbf{R}([\mathbf{u}_\tau]) \tag{2.4}$$

where $0 < \gamma_\tau$ is an experimentally determined constant characterizing the shear stiffness of the adhesive seam and that the shear operator $\mathbf{R}(\mathbf{v})$ acts according to the formula

$$\mathbf{R}(\mathbf{v}) = \begin{cases} \mathbf{v}, & |\mathbf{v}| \leq L \\ L|\mathbf{v}|^{-1}\mathbf{v}, & |\mathbf{v}| \geq L \end{cases} \tag{2.5}$$

In writing down relation (2.4), it has been assumed that the friction forces (which can arise when $[u_3] = 0$) can be neglected compared with the tangential reaction of the adhesive seam.

We will make the following assumptions with regard to the stiffnesses γ_n and γ_τ and the characteristic length L

$$\gamma_n = \varepsilon^{-1}\gamma_n^*, \quad \gamma_\tau = \varepsilon^{-1}\gamma_\tau^*, \quad L = \varepsilon L^* \tag{2.6}$$

where the quantities γ_n^*, γ_τ^* and L^* are independent of the parameter ε. Note that the meaning of the last assumption is already clear: the characteristic length of the adhesive bonds L is comparable with the thickness of the layers h.

3. Application of the averaging method

The stress-strain state of the layers is defined by the system of equations of the linear theory of elasticity (henceforth (i, j = 1, 2, 3))

$$\sum_{i=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j} = 0 \tag{3.1}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}), \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{3.2}$$

Here, λ and μ are the Lamé parameters and δ_{ij} is the Kronecker delta.

The displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ within each layer satisfies the Lamé equation

$$L(\nabla_x)\mathbf{u}(\mathbf{x}) = 0 \tag{3.3}$$

where $L(\nabla_x)$ is the matrix differential operator which is obtained by substituting expressions (3.2) into Eq. (3.1).

In formulating the boundary value on the deformation of a packet of layers, boundary conditions are imposed on the faces $x_3 = 0$ and $x_3 = H$ and specific conditions are formulated at infinity.

We emphasize that, in spite of the linearity of the initial relations (3.1) and (3.2), the boundary conditions at the interfaces of the layers are intrinsically non-linear, which ultimately imparts a non-linear character to the behaviour of the solution of the averaged equations.

Following the method of averaging (see Refs 16 and 17, for example), we will seek a solution of the problem in the form of the expansion

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \varepsilon\mathbf{u}^1(\mathbf{x},\zeta) + \varepsilon^2\mathbf{u}^2(\mathbf{x},\zeta) + \ldots; \quad \zeta = (\varepsilon H)^{-1}x_3 \tag{3.4}$$

where the dimensionless "fast" variable ζ has been introduced. At the same time, it is assumed that the vector functions $\mathbf{u}^k(\mathbf{x},\zeta)(k = 1,2,\ldots)$ are one-periodic functions with respect to the variable ζ. Starting from physical considerations, it follows that me must assume that the dependence of the vector functions $\mathbf{u}^k(\mathbf{x},\zeta)$ on the variable ζ is smooth everywhere apart from in the interfacial planes of the layers $\zeta = (\varepsilon H)^{-1}x_3^{(s)}$ where they can undergo discontinuities of the first kind.

We substitute expansion (3.4) into Eq. (3.3) and use the formula

$$\frac{d}{dx_3}F\left(\mathbf{x},\frac{x_3}{\varepsilon H}\right) = \left.\left(\frac{\partial F(\mathbf{x},\zeta)}{\partial x_3} + (\varepsilon H)^{-1}\frac{\partial F(\mathbf{x},\zeta)}{\partial\zeta}\right)\right|_{\zeta=(\varepsilon H)^{-1}x_3} \tag{3.5}$$

for the differentiation of a complex function.

As a result, we obtain the following expansion

$$\varepsilon^{-2}L_0\mathbf{u}^0 + \varepsilon^{-1}(L_1\mathbf{u}^0 + L_0\mathbf{u}^1) + \varepsilon^0(L_2\mathbf{u}^0 + L_1\mathbf{u}^1 + L_0\mathbf{u}^2) + \ldots = 0 \tag{3.6}$$

where $L_2(\nabla_x) = L(\nabla_x)$ and we have introduced the notation

$$L_0(\partial_\zeta) = H^{-2}\operatorname{diag}\{\mu\partial_\zeta^2, \mu\partial_\zeta^2, (2\mu+\lambda)\partial_\zeta^2\}, \quad L_1(\nabla_x,\partial_\zeta) = H^{-1}\partial_\zeta l_1(\nabla_x)$$
$$l_1(\nabla_x) = \begin{Vmatrix} 2\mu\partial_3 & 0 & (\lambda+\mu)\partial_1 \\ 0 & 2\mu\partial_3 & (\lambda+\mu)\partial_2 \\ (\lambda+\mu)\partial_1 & (\lambda+\mu)\partial_2 & 2(\lambda+2\mu)\partial_3 \end{Vmatrix}, \quad \partial_i = \frac{\partial}{\partial x_i} \tag{3.7}$$

We require that terms of the orders ε^{-1} and ε^0 in expansion (3.6) should vanish (the term of order ε^{-2} vanishes identically by virtue of the fact that the vector $\mathbf{u}^0(\mathbf{x})$ is independent of the fast variable ζ). In this way, we obtain the following system of differential equations in ζ

$$L_0(\partial_\zeta)\mathbf{u}^1(\mathbf{x},\zeta) = 0 \tag{3.8}$$

$$L_0(\partial_\zeta)\mathbf{u}^2(\mathbf{x},\zeta) = -L(\nabla_x)\mathbf{u}^0(\mathbf{x}) - L_1(\nabla_x,\partial_\zeta)\mathbf{u}^1(\mathbf{x},\zeta) \tag{3.9}$$

for the recurrent determination of lowest terms of expansion (3.4).

Following Nikitin's approach[4], we consider Eqs. (3.8) and (3.9) in a periodicity cell $-1/2 \le \zeta \le 1/2$ (that is, $x_3^{(s)} - \varepsilon H/2 \le x_3 \le x_3^{(s)} + \varepsilon H/2$), containing the interface of the layers. Here, the condition for the 1-periodicity of the lowest terms of expansion (3.4) takes the form

$$\mathbf{u}^k\big|_{\zeta=-1/2} = \mathbf{u}^k\big|_{\zeta=-1/2}, \quad \left.\frac{\partial\mathbf{u}^k}{\partial\zeta}\right|_{\zeta=-1/2} = \left.\frac{\partial u^k}{\partial\zeta}\right|_{\zeta=1/2}; \quad k = 1,2,\ldots \tag{3.10}$$

Using formula (3.5) again, we obtain

$$\sigma_{33} = \sigma_{33}^0 + (2\mu+\lambda)H^{-1}\frac{\partial u_3^1}{\partial\zeta} + \varepsilon\left((2\mu+\lambda)H^{-1}\frac{\partial u_3^2}{\partial\zeta} + \sigma_{33}^1\right) + \dots \quad (3.11)$$

$$\sigma_{3\alpha} = \sigma_{3\alpha}^0 + \mu H^{-1}\frac{\partial u_\alpha^1}{\partial\zeta} + \varepsilon\left(\mu H^{-1}\frac{\partial u_\alpha^2}{\partial\zeta} + \sigma_{3\alpha}^1\right) + \dots \quad (3.12)$$

Here σ_{ij}^k are the stresses corresponding to the vector $\mathbf{u}^k(\mathbf{x},\zeta)(k=0,1,\dots)$ which are calculated using formulae (3.2).
Then, according to expansion (3.4), we shall obviously have

$$[u_3] = \varepsilon[u_3^1] + \varepsilon^2[u_3^2] + \dots \quad (3.13)$$

$$[\mathbf{u}_\tau] = \varepsilon[\mathbf{u}_\tau^1] + \varepsilon^2[\mathbf{u}_\tau^2] + \dots \quad (3.14)$$

In this case (in view of expansions (3.11) and (3.12)), the static equilibrium conditions (1.9) give

$$\left[\frac{\partial u_i^1}{\partial\zeta}\right] = 0, \quad \zeta = 0 \quad (3.15)$$

The solution of Eq. (3.8) has the form

$$u_i^1(\mathbf{x},\zeta) = \begin{cases} \varphi_i^+\zeta + \psi_i^+, & \zeta\in(0,1/2] \\ \varphi_i^-\zeta + \psi_i^-, & \zeta\in[-1/2,0) \end{cases} \quad (3.16)$$

Subjecting the function (3.16) to periodicity conditions (3.10), we obtain

$$\varphi_i^+ = \varphi_i^- = \varphi_i, \quad \psi_i^- - \psi_i^+ = \varphi_i \quad (3.17)$$

Here, conditions (3.16) are automatically satisfied.
So, according to relations (3.17), we shall have

$$[u_i^1] = -\varphi_i, \quad \partial_\zeta u_i^1 = \varphi_i, \quad \zeta = 0 \quad (3.18)$$

With the aim of deriving the conditions for the solvability of Eq. (3.9) in the class of 1-periodic functions, we integrate it with respect to the variable ζ within the limits from – 1/2 to 1/2. On the one hand, on the basis of relations (3.10) we obtain

$$\int_{-1/2}^{1/2}\frac{\partial^2 u_i^2}{\partial\zeta^2}d\zeta = -\left[\frac{\partial u_i^2}{\partial\zeta}\right] \quad (3.19)$$

On the other hand, according to the definition of the differential operator $L_1(\nabla_x,\partial_\zeta)$, we shall have

$$L_1(\nabla_x,\partial_\zeta)\mathbf{u}^1(\mathbf{x},\zeta) = H^{-1}l_1(\nabla_x)\boldsymbol{\varphi}(\mathbf{x}) \quad (3.20)$$

Furthermore, according to continuity conditions (1.9), the equalities

$$[\partial_\zeta u_\alpha^2] = -\mu^{-1}H[\sigma_{3\alpha}^1], \quad \alpha=1,2, \quad [\partial_\zeta u_3^2] = -(2\mu+\lambda)^{-1}H[\sigma_{33}^1] \quad (3.21)$$

must be satisfied, where, when account is taken of relations (3.17),

$$[\sigma_{3\alpha}^1] = -\mu(\partial_3\varphi_\alpha + \partial_\alpha\varphi_3), \quad [\sigma_{33}^1] = -(2\mu+\lambda)\partial_3\varphi_3 - \lambda(\partial_1\varphi_1 + \partial_2\varphi_2) \quad (3.22)$$

Hence, taking relations (3.19) - (3.22) into account, we arrive at the equation

$$L(\nabla_x)\mathbf{u}^0(\mathbf{x}) + H^{-1}l(\nabla_x)\boldsymbol{\varphi}(\mathbf{x}) = 0 \quad (3.23)$$

where the following notation has been introduced

$$(\nabla_x) = \begin{Vmatrix} \mu\partial_3 & 0 & \lambda\partial_1 \\ 0 & \mu\partial_3 & \lambda\partial_2 \\ \mu\partial_1 & \mu\partial_2 & (2\mu+\lambda)\partial_3 \end{Vmatrix} \quad (3.24)$$

We now return to the boundary conditions on the contact surface.

4. Solution of the problem in a periodicity cell in the case of rough layers

Substituting expansions (3.13) and (3.11) into boundary condition (1.3) and taking account of relation (1.11), after taking the limit when $\varepsilon \to 0$, we obtain

$$\sigma_{33}^0 + (2\mu + \lambda)H^{-1}\partial_\zeta u_3^1 = -c_m^*(-[u_3^1])_+^m, \quad \zeta = 0 \tag{4.1}$$

According to boundary condition (1.6), in the cohesion zone $\Gamma_{ca}^{(s)}$ we have ($\zeta = 0$)

$$[u_3^1] < 0, \quad [u_1^1] = [u_2^1] = 0, \quad \left(\sum(\mu H^{-1}\partial_\zeta u_\alpha^1 + \sigma_{3\alpha}^0)^2\right)^{1/2} < fc_m^*(-[u_3^1])^m \tag{4.2}$$

Summation is henceforth carried out from $\alpha = 1$ to $\alpha = 2$.

Assuming that $|[\boldsymbol{u}_\tau]| \equiv ([u_1^1]^2 + [u_2^1]^2)^{1/2} \neq 0$, in the slippage zone $\Gamma_{cs}^{(s)}$, according to boundary condition (1.7), we obtain ($\zeta = 0$)

$$[u_3^1] < 0, \quad \left(\sum(\mu H^{-1}\partial_\zeta u_\alpha^1 + \sigma_{3\alpha}^0)^2\right)^{1/2} = fc_m^*(-[u_3^1])^m$$
$$\frac{\mu H^{-1}\partial_\zeta u_\beta^1 + \sigma_{3\beta}^0}{\left(\sum(\mu H^{-1}\partial_\zeta u_\alpha^1 + \sigma_{3\alpha}^0)^2\right)^{1/2}} = \frac{[u_\beta^1]}{\left(\sum[u_\alpha^1]^2\right)^{1/2}}, \quad \beta = 1, 2 \tag{4.3}$$

Taking account of relation (3.18), we impart the form

$$\sigma_{33}^0 + (2\mu + \lambda)H^{-1}\varphi_3 = -c_m^*(\varphi_3)_+^m$$

to boundary condition (4.1).

From this we derive the relations

$$\sigma_{33}^0 \geq 0 \Rightarrow \varphi_3 = -(2\mu + \lambda)^{-1}H\sigma_{33}^0$$
$$\sigma_{33}^0 < 0 \Rightarrow (2\mu + \lambda)H^{-1}\varphi_3 + c_m^*\varphi_3^m = -\sigma_{33}^0 \tag{4.4}$$

for determining the function φ_3.

When account is taken of relations (3.18), boundary condition (4.2) can be rewritten as

$$\varphi_3 > 0, \quad \left|\boldsymbol{\sigma}_\tau^0\right| < fc_m^*\varphi_3^m \Rightarrow \varphi_1 = \varphi_2 = 0 \tag{4.5}$$

Taking account of boundary condition (4.3), we write boundary condition (4.3) in the form

$$\varphi_3 > 0, \quad \left(\sum(\mu H^{-1}\varphi_\alpha + \sigma_{3\alpha}^0)^2\right)^{1/2} = fc_m^*\varphi_3^m$$
$$\frac{\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau}{\left|\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau\right|} = \frac{\boldsymbol{\varphi}_\tau}{|\boldsymbol{\varphi}_\tau|}; \quad \boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3), \quad \boldsymbol{\varphi}_\tau = (\varphi_1, \varphi_2, 0) \tag{4.6}$$

whence, when account is taken of the first equality of (4.6), we obtain the expression

$$\boldsymbol{\varphi}_\tau = -\frac{|\boldsymbol{\varphi}_\tau|}{fc_m^*\varphi_3^m + \mu H^{-1}|\boldsymbol{\varphi}_\tau|}\boldsymbol{\sigma}_\tau^0 \tag{4.7}$$

and, substituting this into the second equality of (4.6) and taking account of the inequality $\varphi_3 > 0$, we obtain

$$\left|\boldsymbol{\sigma}_\tau^0\right| = fc_m^*\varphi_3^m + \mu H^{-1}|\boldsymbol{\varphi}_\tau| \tag{4.8}$$

Expressing the quantity $|\varphi_\tau|$ from Eq. (4.8) and substituting the result into formula (4.7), we establish the relation

$$\boldsymbol{\varphi}_\tau = -\frac{\left|\boldsymbol{\sigma}_\tau^0\right| - fc_m^*\varphi_3^m}{\mu H^{-1}}\frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|}$$

We recall that formula (4.9) was derived from relations (4.6) or, what is the same thing, from (4.3) for the condition $|[\boldsymbol{u}_\tau^1]| \neq 0$, that is, for the condition $|\boldsymbol{\varphi}_\tau| \neq 0$. Equality (4.8) enables this condition to be replaced by the inequality $|\sigma_\tau^0| > fc_m^*\varphi_m^3$.

Finally, when account is taken of expansions (3.11) - (3.13) and relations (3.18), the condition for there to be no contact (1.5) gives

$$\varphi_3 < 0 \Rightarrow \sigma_{33}^0 + (2\mu + \lambda)H^{-1}\varphi_3 = 0$$
$$\sigma_{3\alpha}^0 + \mu H^{-1}\varphi_\alpha = 0, \quad \alpha = 1, 2 \tag{4.9}$$

Hence, on the basis of relations (4.4), (4.5), (4.8) and (4.10), the problem of determining the functions $\varphi_3 < 0$ and $\boldsymbol{\sigma}_\tau$ can be formulated as follows:

$$\sigma_{33}^0 \geq 0 \Rightarrow \varphi_3 = -(2\mu+\lambda)^{-1} H\sigma_{33}^0, \quad \boldsymbol{\varphi}_\tau = -\mu^{-1} H\boldsymbol{\sigma}_\tau^0 \tag{4.10}$$

$$\sigma_{33}^0 < 0 \Rightarrow (2\mu+\lambda)H^{-1}\varphi_3 + c_m^*\varphi_3^m = -\sigma_{33}^0 \tag{4.11}$$

$$\varphi_3 > 0, \quad \left|\boldsymbol{\sigma}_\tau^0\right| < fc_m^*\varphi_3^m \Rightarrow \boldsymbol{\varphi}_\tau = 0 \tag{4.12}$$

$$\varphi_3 > 0, \quad \left|\boldsymbol{\sigma}_\tau^0\right| \geq fc_m^*\varphi_3^m \Rightarrow \boldsymbol{\varphi}_\tau = -\frac{\left|\boldsymbol{\sigma}_\tau^0\right| - fc_m^*\varphi_3^m}{\mu H^{-1}} \frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \tag{4.13}$$

Note that Eq. (4.12) has a unique solution when $m > 0$.

5. Solution of the problem for a periodicity cell in the case when there is adhesion

Substituting expansions (3.13) and (3.11) into boundary conditions (2.3) and (2.4) and taking account of relations (2.6) and (3.18), we obtain

$$\varphi_3 = 0 \Rightarrow \sigma_{33}^0 + (2\mu+\lambda)H^{-1}\varphi_3 \leq 0$$

$$\varphi_3 < 0 \Rightarrow \sigma_{33}^0 + (2\mu+\lambda)H^{-1}\varphi_3 - \gamma_n^*\beta^2 R^*(-\varphi_3) = 0$$

$$\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau = -\gamma_\tau^*\beta^2 \mathbf{R}^*(\boldsymbol{\varphi}_\tau) \tag{5.1}$$

The shear functions $R^*(\upsilon)$ and $\mathbf{R}^*(\mathbf{v})$ are defined by formulae (2.2) and (2.5), in which the parameter L is replaced by L^*.

Considering the first two relations of (5.1) as a problem in the function φ_3, we obtain

$$\sigma_{33}^0 \leq 0 \Rightarrow \varphi_3 = 0$$

$$0 < \sigma_{33}^0 \leq ((2\mu+\lambda)H^{-1} + \gamma_n^*\beta^2)L^* \Rightarrow -\varphi_3 = ((2\mu+\lambda)H^{-1} + \gamma_n^*\beta^2)^{-1}\sigma_{33}^0$$

$$\sigma_{33}^0 > ((2\mu+\lambda)H^{-1} + \gamma_n^*\beta^2)L^* \Rightarrow -\varphi_3 = (2\mu+\lambda)^{-1}H(\sigma_{33}^0 - \gamma_n^*\beta^2 L^*) \tag{5.2}$$

The last equation of (5.1) is equivalent to the system of relations

$$\left|\boldsymbol{\varphi}_\tau\right| \leq L^* \Rightarrow \boldsymbol{\sigma}_\tau^0 = -\mu H^{-1}\boldsymbol{\varphi}_\tau - \gamma_\tau^*\beta^2\boldsymbol{\varphi}_\tau \tag{5.3}$$

$$\left|\boldsymbol{\varphi}_\tau\right| \geq L^* \Rightarrow \boldsymbol{\sigma}_\tau^0 = -\mu H^{-1}\boldsymbol{\varphi}_\tau - \gamma_\tau^*\beta^2 L^* \frac{\boldsymbol{\varphi}_\tau}{\left|\boldsymbol{\varphi}_\tau\right|} \tag{5.4}$$

The solution of Eq. (5.3) gives

$$\left|\boldsymbol{\sigma}_\tau^0\right| \leq (\mu H^{-1} + \gamma_\tau^*\beta^2)L^* \Rightarrow \boldsymbol{\varphi}_\tau = -(\mu H^{-1} + \gamma_\tau^*\beta^2)^{-1}\boldsymbol{\sigma}_\tau^0 \tag{5.5}$$

From Eq. (5.4), we obtain

$$\boldsymbol{\varphi}_\tau = -(\mu H^{-1} + \gamma_\tau^*\beta^2 L^* \left|\boldsymbol{\varphi}_\tau\right|^{-1})^{-1}\boldsymbol{\sigma}_\tau^0, \quad \left|\boldsymbol{\varphi}_\tau\right| = \mu H^{-1}\left(\left|\boldsymbol{\sigma}_\tau^0\right| - \gamma_\tau^*\beta^2 L^*\right) \tag{5.6}$$

Substituting the second expression of (5.6) into the first formula of (5.6), we find

$$\boldsymbol{\varphi}_\tau = -\mu H^{-1}\left(\left|\boldsymbol{\sigma}_\tau^0\right| - \gamma_\tau^*\beta^2 L^*\right)\frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \tag{5.7}$$

According to inequality (5.4), formula (5.7) will only be correct when the following condition is satisfied

$$\mu H^{-1}\left(\left|\boldsymbol{\sigma}_\tau^0\right| - \gamma_\tau^*\beta^2 L^*\right) \geq L^*$$

We shall therefore have

$$\left|\boldsymbol{\sigma}_\tau^0\right| \geq (\mu H^{-1} + \gamma_\tau^*\beta^2)L^* \Rightarrow \boldsymbol{\varphi}_\tau = -\mu H^{-1}\left(\left|\boldsymbol{\sigma}_\tau^0\right| - \gamma_\tau^*\beta^2 L^*\right)\frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \tag{5.8}$$

The differential equation

$$\dot{\beta} = -\varepsilon^2\gamma_{\rm de}(R^*(-\varphi_3))^2\beta \tag{5.9}$$

which follows from (2.1) with the initial condition

$$\beta(0,\mathbf{x})=\beta_0(\mathbf{x}) \quad (5.10)$$

is added to relations (5.5), (5.8) and (5.2).

The averaged cohesion function $\beta(t,\mathbf{x})$ depends on the time-like parameter t and the coordinates $\mathbf{x}$ and, for simplicity, is denoted by the same symbol as the surface cohesion function.

6. Governing relations of a laminar medium with rough layers

Following the well known approach,[2] we consider a laminated medium filling the whole of space. Suppose it is subjected to a homogeneous forced deformation $\varepsilon_{ij}^0=\text{const}(i,j=1,2,3)$ corresponding to a linear displacement vector function $\mathbf{u}^0(\mathbf{x})$.

The expansion of the components of the stress tensor (3.11), (3.12) corresponds to the expansion of the displacement vector (3.4). It is easy to write out similar expansions for the strain tensor components.

We introduce the averaged stresses and strains

$$\tilde{\sigma}_{ij}=\int_{-1/2}^{1/2}\sigma_{ij}\,d\zeta,\quad \tilde{\varepsilon}_{ij}=\int_{-1/2}^{1/2}\varepsilon_{ij}\,d\zeta$$

When account is taken of relations (3.18), we shall have (everywhere henceforth $\alpha,\beta=1,2$)

$$\tilde{\sigma}_{33}=\sigma_{33}^0+(2\mu+\lambda)H^{-1}\varphi_3,\quad \tilde{\sigma}_{3\alpha}=\sigma_{3\alpha}^0+\mu H^{-1}\varphi_\alpha$$
$$\tilde{\sigma}_{\alpha\alpha}=\sigma_{\alpha\alpha}^0+\lambda H^{-1}\varphi_3,\quad \tilde{\sigma}_{\alpha\beta}=\sigma_{\alpha\beta}^0 \quad (6.1)$$

$$\tilde{\varepsilon}_{33}=\varepsilon_{33}^0+H^{-1}\varphi_3,\quad \tilde{\varepsilon}_{3\alpha}=\varepsilon_{3\alpha}^0+(2H)^{-1}\varphi_\alpha,\quad \tilde{\varepsilon}_{\alpha\beta}=\varepsilon_{\alpha\beta}^0 \quad (6.2)$$

These formulae can be represented in the form

$$\tilde{\varepsilon}_{ij}=\varepsilon_{ij}^0+\varepsilon_{ij}^*,\quad \varepsilon_{\alpha\beta}^*=0,\quad \varepsilon_{3\alpha}^*=(2H)^{-1}\varphi_\alpha,\quad \varepsilon_{33}^*=H^{-1}\varphi_3 \quad (6.3)$$

$$\tilde{\sigma}_{ij}=\sigma_{ij}^0+\sigma_{ij}^*;\quad \sigma_{ij}^*=2\mu\varepsilon_{ij}^*+\lambda\delta_{ij}(\varepsilon_{11}^*+\varepsilon_{22}^*+\varepsilon_{33}^*) \quad (6.4)$$

Note that, in the notation of averaged stresses (6.1), the resulting equation (3.23), when account is taken of equality (3.24), can be rewritten in the usual form (3.1).

Taking account of formulae (6.3) and (6.4), the previously obtained solution (4.11) - (4.14) of the problem in a periodicity cell can be represented as follows:

$$\sigma_{33}^0\ge 0\Rightarrow\tilde{\sigma}_{33}=\tilde{\sigma}_{31}=\tilde{\sigma}_{32}=0 \quad (6.5)$$

$$\sigma_{33}^0<0\Rightarrow(2\mu+\lambda)\varepsilon_{33}^*+c_m^*H^m(\varepsilon_{33}^*)^m=-\sigma_{33}^0 \quad (6.6)$$

$$\varepsilon_{33}^*>0,\quad \left|\boldsymbol{\sigma}_\tau^0\right|<fc_m^*H^m(\varepsilon_{33}^*)^m\Rightarrow\varepsilon_{31}^*=\varepsilon_{32}^*=0,\quad \tilde{\boldsymbol{\sigma}}_\tau=\boldsymbol{\sigma}_\tau^0$$
$$\varepsilon_{33}^*>0,\quad \left|\boldsymbol{\sigma}_\tau^0\right|\ge fc_m^*H^m(\varepsilon_{33}^*)^m\Rightarrow\tilde{\boldsymbol{\sigma}}_\tau=fc_m^*(\varepsilon_{33}^*)^m\frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \quad (6.7)$$

$$\varepsilon_{3\alpha}^*=-(2\mu)^{-1}\left(\left|\boldsymbol{\sigma}_\tau^0\right|-fc_m^*H^m(\varepsilon_{33}^*)^m\right)\frac{\sigma_{3\alpha}^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \quad (6.8)$$

At the same time, the second relation of (6.3) and relations (6.4) are satisfied.

In the case of a large stiffness c_m^* on the left-hand side of Eq. (6.6), the first term can be neglected and relations (6.6)- (6.8) can be simplified as

$$\sigma_{33}^0<0\Rightarrow\varepsilon_{33}^*=(c_m^*)^{-1/m}H^{-1}\left|\sigma_{33}^0\right|^{1/m} \quad (6.9)$$

$$\sigma_{33}^0<0,\quad \left|\boldsymbol{\sigma}_\tau^0\right|<f\left|\sigma_{33}^0\right|\Rightarrow\varepsilon_{31}^*=\varepsilon_{32}^*=0,\quad \tilde{\boldsymbol{\sigma}}_\tau=\boldsymbol{\sigma}_\tau^0$$
$$\sigma_{33}^0<0,\quad \left|\boldsymbol{\sigma}_\tau^0\right|\ge f\sigma_{33}^0\Rightarrow\tilde{\boldsymbol{\sigma}}_\tau=f\left|\sigma_{33}^0\right|\frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \quad (6.10)$$

$$\varepsilon_{3\alpha}^*=-(2\mu)^{-1}\left(\left|\boldsymbol{\sigma}_\tau^0\right|-f\left|\sigma_{33}^0\right|\right)\frac{\sigma_{3\alpha}^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \quad (6.11)$$

Here, the second relation of (6.3) and relations (6.4) and (6.5) are satisfied.

Finally, we take the limit as $c_m^*\to+\infty$. Relations (6.10) and (6.11) obviously remain unchanged while relation (6.9) takes the form

$$\sigma_{33}^0<0\Rightarrow\varepsilon_{33}^*=0,\quad \tilde{\sigma}_{33}=\sigma_{33}^0 \quad (6.12)$$

The first relation of (6.3) and relations (6.4), (6.5), (6.10) - (6.12) are identical, except for the form in which they are written, with the previously obtained solution[2] in the case of a laminated medium, the layers of which are in unilateral contact with Coulomb friction.

Remark. The following law has been proposed[24] for describing the friction of plastic bodies

$$|\boldsymbol{\sigma}_\tau| \leq \min\{f|\sigma_{33}|, \tau_s\} \tag{6.13}$$

where τ_s is the maximum shear stress permitted by the flow condition adopted (for the Teresa flow condition and the Mises flow condition in the case of plane flow, the quantity τ_s is identical to the yield limit under pure shear and is a constant of the material).

The equality in (6.13) holds in the case of relative slippage and the inequality holds in the case of relative quiescence of the contacting bodies. The introduction of condition (6.13) is explained by the fact that, in the domain of plastic deformations, the normal pressure is large and usually $f|\sigma_{33}| > \tau_s$.

Assuming that friction law (6.13) approximately characterizes the case of elastoplastic shear deformation of microroughnesses, we now consider the possibility of generalizing the solution (6.3), (6.4) - (6.8) obtained above.

Thus, instead of boundary condition (1.4), we have the following

$$|\boldsymbol{\sigma}_\tau| \leq \min\{f(-\sigma_{33})_+, \tau_s\}$$
$$(|\boldsymbol{\sigma}_\tau| - \min\{f(-\sigma_{33})_+, \tau_s\})[\mathbf{u}_\tau] = 0, \quad \boldsymbol{\sigma}_\tau \cdot [\mathbf{u}_\tau] = |\boldsymbol{\sigma}_\tau||\mathbf{u}_\tau| \tag{6.14}$$

In the cohesion zone, instead of relations (4.2), on the basis of boundary condition (6.14) we obtain ($\zeta = 0$)

$$[u_3^1] < 0, \quad [u_1^1] = [u_2^1] = 0$$

$$\left(\sum(\mu H^{-1}\partial_\zeta u_\alpha^1 + \sigma_{3\alpha}^0)^2\right)^{1/2} < \min\{fc_m^*(-[u_3^1])^m, \tau_s\}$$

from which, when account is taken of relations (3.18), we obtain

$$\varphi_3 > 0, \quad |\boldsymbol{\sigma}_\tau^0| < M_1 \Rightarrow \boldsymbol{\varphi}_\tau = 0; \quad M_1 = \min\{fc_m^*\varphi_3^m, \tau_s\} \tag{6.15}$$

In the slippage zone, instead of relations (4.6) we shall have

$$\varphi_3 > 0, \quad |\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau| = M_1, \quad \frac{\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau}{|\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau|} = -\frac{\boldsymbol{\varphi}_\tau}{|\boldsymbol{\varphi}_\tau|}$$

Hence, in a similar way to operations on (4.7) - (4.9), we have

$$\varphi_3 > 0, \quad |\boldsymbol{\sigma}_\tau^0| \geq M_1 \Rightarrow \boldsymbol{\varphi}_\tau = -\frac{|\boldsymbol{\sigma}_\tau^0| - M_1}{\mu H^{-1}} \frac{\boldsymbol{\sigma}_\tau^0}{|\boldsymbol{\sigma}_\tau^0|} \tag{6.16}$$

Finally, we rewrite relations (6.15) and (6.16) in terms of the quantities $\tilde{\sigma}_{ij}$ and ε_{ij}^* as follows:

$$\varepsilon_{33}^* > 0, \quad |\boldsymbol{\sigma}_\tau^0| < M_2 \Rightarrow \varepsilon_{31}^* = \varepsilon_{32}^* = 0, \quad \tilde{\boldsymbol{\sigma}}_\tau = \boldsymbol{\sigma}_\tau^0$$
$$\varepsilon_{33}^* > 0, \quad |\boldsymbol{\sigma}_\tau^0| \geq M_2 \Rightarrow \tilde{\boldsymbol{\sigma}}_\tau = M_2\frac{\boldsymbol{\sigma}_\tau^0}{|\boldsymbol{\sigma}_\tau^0|} \tag{6.17}$$

$$\varepsilon_{3\alpha}^* = -(2\mu)^{-1}\left(|\boldsymbol{\sigma}_\tau^0| - M_2\right)\frac{\sigma_{3\alpha}^0}{|\boldsymbol{\sigma}_\tau^0|} \tag{6.18}$$

where

$$M_2 = \min\{fc_m^*\varepsilon_{33}^*, \tau_s\}$$

Here, the second relation of (6.3) and relations (6.4) - (6.6) are added to relations (6.17) and (6.18).

7. The contact of dry surfaces with adhesion and friction

An attractive molecular force arises between them when smooth surfaces approach one another. We will formulate the boundary conditions which simulate the interaction of the surfaces of the elastic layers taking account of adhesion and Coulomb friction.

Firstly, we shall assume that the contact is unilateral, that is,

$$[u_3] \geq 0 \tag{7.1}$$

Secondly, the adhesive forces only act at a certain distance, that is, we put

$$[u_3] \geq h_0 \Rightarrow \sigma_{33} = \sigma_{31} = \sigma_{32} = 0 \tag{7.2}$$

Thirdly, following the approach in Ref. 25, we shall initially assume that the adhesive forces are constant and

$$0 < [u_3] < h_0 \Rightarrow \sigma_{33} = p_0, \quad \sigma_{31} = \sigma_{32} = 0 \tag{7.3}$$

Fourthly, in the case of complete contact, when $[u_3]=0$, the adhesive forces are replaced by elastic contact interaction forces with Coulomb friction:

$$[u_3]=0 \Rightarrow \sigma_{33} \le 0, \quad |\boldsymbol{\sigma}_\tau| + f\sigma_{33} \le 0, \quad (|\boldsymbol{\sigma}_\tau| + f\sigma_{33})[\mathbf{u}_\tau] = 0, \quad \boldsymbol{\sigma}_\tau \cdot [\mathbf{u}_\tau] = |\boldsymbol{\sigma}_\tau||[\mathbf{u}_\tau]| \tag{7.4}$$

The constants p_0 and h_0 have the meaning of an adhesive force stress and the maximum distance at which attractive molecular forces act. The relation $p_0h_0=\gamma$ (see Ref. 10, for example) holds, where γ is the surface energy density.

With respect to the quantity h_0, we make the assumption

$$h_0 = \varepsilon h_0^* \tag{7.5}$$

where the quantity h_0^* is independent of the parameter ε.

When account is taken of Eq. (7.5) and relations (3.18), relations (7.1)- (7.3) lead to the following problem in the function φ_3.

$$\varphi_3 \le 0 \tag{7.6}$$

$$-\varphi_3 \ge h_0^* \Rightarrow \sigma_{33}^0 + (2\mu+\lambda)H^{-1}\varphi_3 = 0, \quad \mu H^{-1}\varphi_\alpha + \sigma_{3\alpha}^0 = 0 \tag{7.7}$$

$$0 < -\varphi_3 < h_0^* \Rightarrow \sigma_{33}^0 + (2\mu+\lambda)H^{-1}\varphi_3 = p_0, \quad \mu H^{-1}\varphi_\alpha + \sigma_{3\alpha}^0 = 0 \tag{7.8}$$

The following immediately ensues from relation (7.7)

$$\sigma_{33}^0 \ge (2\mu+\lambda)H^{-1}h_0^* \Rightarrow \varphi_3 = -(2\mu+\lambda)^{-1}H\sigma_{33}^0, \quad \varphi_\alpha = -\mu^{-1}H\sigma_{3\alpha}^0 \tag{7.9}$$

Expressing the quantity φ_3 from the first equation of (7.8), we obtain

$$\varphi_3 = -(2\mu+\lambda)^{-1}H(\sigma_{33}^0 - p_0) \tag{7.10}$$

We now introduce the notation

$$p_0^*(h) = p_0 + (2\mu+\lambda)H^{-1}h$$

Substituting the quantity (7.10) into double inequality (7.8), we arrive at the following inequality

$$p_0 < \sigma_{33}^0 < p_0^*(h_0^*) \tag{7.11}$$

Two important facts accompany relations (7.9)- (7.11). First, problem (7.6) - (7.8) does not permit the determination of the quantity φ_3 when the stress σ_{33} lies in the interval $(0,p_0)$. Second, when

$$(2\mu+\lambda)H^{-1}h_0^* \le \sigma_{33}^0 < p_0^*(h_0^*)$$

formulae (7.9) and (7.10) give two different values of the magnitude of φ_3.

The contradictions mentioned are a consequence adhesion law (7.3). This becomes obvious when the following simple example is considered. Suppose two small elastic springs are connected in series using two smooth small plates, between which adhesive forces act according to the law (7.1) - (7.3). It is then found that, when small stresses (which are insufficient to overcome the adhesive forces) are applied to the free ends of the springs, the mechanical system being considered cannot be in equilibrium. We emphasize that the piecewise-constant adhesive force law (7.3) is also a very rough approximation of the initial positive segment of the Lennard–Jones potential curve. The following (continuous) piecewise-linear dependence enables us to obtain more accurate results (both quantitatively and qualitatively)

$$0 < [u_3] < h_1 \Rightarrow \sigma_{33} = p_0\frac{[u_3]}{h_1}, \quad \sigma_{31} = \sigma_{32} = 0 \tag{7.12}$$

$$h_1 \le [u_3] \le h_0 \Rightarrow \sigma_{33} = p_0\frac{h_0 - [u_3]}{h_0 - h_1}, \quad \sigma_{31} = \sigma_{32} = 0 \tag{7.13}$$

With respect to the quantities $0<h_1<h_0$, we make the assumption

$$h_1 = \varepsilon h_1^*, \quad h_0 = \varepsilon h_0^* \tag{7.14}$$

where the quantities h_1^* and h_0^* are independent of the parameter ε.

Remark. The choice of the parameters p_0,h_0 and h_1, appearing in the proposed law of adhesion (7.1), (7.2), (7.12) and (7.13), remains an open question to some extent. In particular, the parameters p_0 and h_1 can be determined from the maximum of the Lennard–Jones potential curve. At the same time, following the well known approach,[10] energy considerations can be invoked for determining the value of h_0. In this case (irrespective of the dependence on the value of the parameter $h_1 \in (0,h_0)$), we shall have

$$p_0h_0 = 2\gamma$$

Note that the value of h_0, calculated using this formula, gives a width of the zone of action of the adhesive forces which is twice as large compared with the value of h_0 occurring in the piecewise-constant approximation (7.3) for the same values of the adhesive force stress p_0.

8. The solution of the problem in a periodicity cell in the case of the contact of dry surfaces with adhesion and friction

As before, from relations (7.1), (7.2), (7.12) and (7.13), taking account of relations (3.18) and (7.14), we derive the problem of determining the function φ_3. We have relations (7.6), (7.7) and the following

$$0 < -\varphi_3 \le h_1^* \Rightarrow \mu H^{-1}\varphi_\alpha + \sigma_{3\alpha}^0 = 0, \quad \sigma_{33}^0 + \frac{p_0^*(h_1^*)\varphi_3}{h_1^*} = 0 \tag{8.1}$$

$$h_1^* \le -\varphi_3 \le h_0^* \Rightarrow \mu H^{-1}\varphi_\alpha + \sigma_{3\alpha}^0 = 0, \quad \sigma_{33}^0 = \frac{p_0^* h_1^* - h_0^*}{h_1^* - h_0^*}\phi_3 = -\frac{p_0 h_0^*}{h_1^* - h_0^*} \tag{8.2}$$

From relation (8.1), we obtain

$$0 < \sigma_{33}^0 < p_0^*(h_1^*) \Rightarrow \varphi_\alpha = -\mu^{-1}H\sigma_{3\alpha}^0, \quad \varphi_3 = -\frac{\sigma_{33}^0 h_1^*}{p_0^*(h_1^*)} \tag{8.3}$$

The result of the solution of relation (8.2) depends on the sign of the quantity $p_0^*(h_1^* - h_0^*)$. It is just in the case of strong adhesion that the solution of the problem in a periodicity cell again turns out to be non-unique. We shall therefore henceforth assume that

$$p_0^*(h_1^* - h_0^*) < 0 \tag{8.4}$$

With assumption (8.4), we then obtain the following from relation (8.2)

$$p_0^*(h_1^*) \le \sigma_{33}^0 < (2\mu + \lambda)H^{-1}h_0^* \Rightarrow \varphi_\alpha = -\mu^{-1}H\sigma_{3\alpha}^0, \quad \varphi_3 = -\frac{\sigma_{33}^0(h_1^* - h_0^*) + p_0 h_0^*}{p_0^*(h_1^* - h_0^*)} \tag{8.5}$$

We now turn to the case of contact defined by condition (7.4) from which, when account is taken of relations (3.8), we derive

$$\begin{gathered}\varphi_3 = 0 \Rightarrow \sigma_{33}^0 \le 0, \quad \left|\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau\right| + f\sigma_{33}^0 \le 0 \\ \left(\left|\boldsymbol{\sigma}_\tau^0\right| + \mu H^{-1}\varphi_\tau\right| + f\sigma_{33}^0\right)[\boldsymbol{\varphi}_\tau] = 0, \quad \frac{\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau}{\left|\boldsymbol{\sigma}_\tau^0 + \mu H^{-1}\boldsymbol{\varphi}_\tau\right|} = -\frac{\boldsymbol{\varphi}_\tau}{|\boldsymbol{\varphi}_\tau|}\end{gathered} \tag{8.6}$$

Considering the system of relations (8.6) as a problem concerning the determination of the vector φ_τ when $\sigma_{33}^0 \le 0$, we obtain

$$\sigma_{33}^0 = 0 \Rightarrow \varphi_3 = 0, \quad \boldsymbol{\varphi}_\tau = -\mu^{-1}H\boldsymbol{\sigma}_\tau^0 \tag{8.7}$$

$$\sigma_{33}^0 < 0, \quad \left|\boldsymbol{\sigma}_\tau^0\right| < f\sigma_{33}^0 \Rightarrow \varphi_3 = 0, \quad \boldsymbol{\varphi}_\tau = 0 \tag{8.8}$$

$$\sigma_{33}^0 < 0, \quad \left|\boldsymbol{\sigma}_\tau^0\right| \ge f\left|\sigma_{33}^0\right| \Rightarrow \varphi_3 = 0, \quad \boldsymbol{\varphi}_\tau = -\frac{\left|\boldsymbol{\sigma}_\tau^0\right| - f\left|\sigma_{33}^0\right|}{\mu H^{-1}}\frac{\boldsymbol{\sigma}_\tau^0}{\left|\boldsymbol{\sigma}_\tau^0\right|} \tag{8.9}$$

Hence, for any stresses $\sigma_{ij}(i,j = 1,2,3)$, the problem in a periodicity cell, corresponding to the piecewise-linear adhesion law (7.12), (7.13), only has a unique solution if the parameters of the model satisfy condition (8.4).

9. Governing relations of the laminated medium with adhesion and friction on the dry contact surfaces of the layers

Consider a laminated medium which fills the whole of space. Suppose it is subjected to a homogeneous forced deformation $\varepsilon_{ij}^0 = \text{const}(i, j = 1, 2, 3)$, corresponding to a linear displacement vector function $\mathbf{u}^0(\mathbf{x})$.

Taking account of relations (6.3) and (6.4), we rewrite the solution (7.9), (8.3), (8.5), (8.7) - (8.9) found earlier in terms of the averaged stresses $\tilde{\sigma}_{ij}$ and the complementary strains $\varepsilon_{ij}^* = \tilde{\varepsilon}_{ij} - \varepsilon_{ij}^0$.

If $\sigma_{33}^0 < 0$, the laminated medium is in a state of transverse compression and the surfaces of neighbouring layers exert a pressure on each another. In this case, relations (8.8) and (8.9) are equivalent to relations (6.10) - (6.12)

When σ_{33}^0, the surfaces of the layers freely slip with respect to one another, that is, according to relation (8.7), we shall have

$$\sigma_{33}^0 = 0 \Rightarrow \varepsilon_{33}^* = 0, \quad \varepsilon_{3\alpha}^* = -\varepsilon_{3\alpha}^0, \quad \tilde{\sigma}_{3i} = 0 \tag{9.1}$$

In the case of a small transverse elongation, according to relation (8.3) we have (everywhere henceforth $\alpha = 1, 2$)

$$\begin{gathered}0 < \sigma_{33}^0 < p_0^*(h_1^*) \Rightarrow \varepsilon_{3\alpha}^* = -\varepsilon_{3\alpha}^0, \quad \tilde{\sigma}_{3\alpha} = 0 \\ \varepsilon_{33}^* = \frac{h_1^*\sigma_{33}^0}{Hp_0^*(h_1^*)}, \quad \tilde{\sigma}_{33} = \frac{p_0\sigma_{33}^0}{p_0^*(h_1^*)}\end{gathered} \tag{9.2}$$

The last formula of (9.2) shows that, over the range of variation of the stress σ_{33}^0 being considered, the average stress $\tilde{\sigma}_{33}$ increases as the value of σ_{33}^0 becomes larger and reaches a maximum value, equal to p_0, at the upper limit of the double inequality (9.2).

Then, according to relations (8.5), we shall have

$$p_0^*(h_1^*) \le \sigma_{33}^0 < (2\mu + \lambda) H^{-1} h_0^* \Rightarrow \varepsilon_{3\alpha}^* = -\varepsilon_{3\alpha}^0, \quad \tilde{\sigma}_{3\alpha} = 0$$
$$\varepsilon_{33}^* = -\frac{(h_1^* - h_0^*)\sigma_{33}^0 + h_0^* p_0}{H p_0^*(h_1^* - h_0^*)}, \quad \tilde{\sigma}_{33} = \frac{p_0(\sigma_{33}^0 - (2\mu + \lambda) H^{-1} h_0^*)}{p_0^*(h_1^* - h_0^*)} \tag{9.3}$$

Hence, when the stresses σ_{33}^0 vary within the limits defined by the double inequality (9.3), then, according to the last formula of (9.3), the average stress $\tilde{\sigma}_{33}$ decreases linearly with respect to σ_{33}^0 from p_0 to zero.

Finally, according to relation (7.9), we obtain

$$(2\mu + \lambda) H^{-1} h_0^* \le \sigma_{33}^0 \Rightarrow \varepsilon_{3\alpha}^* = -\varepsilon_{3\alpha}^0, \quad \tilde{\sigma}_{3\alpha} = 0$$
$$\varepsilon_{33}^* = -(2\mu + \lambda)^{-1}\sigma_{33}^0, \quad \tilde{\sigma}_{33} = 0 \tag{9.4}$$

Formulae (9.4) correspond to the case of a considerable forced transverse stretching deformation of the laminated medium that exceeds the limit at which the adhesive forces are able to keep the layers bonded.

For simplicity, suppose $\varepsilon_{11}^0 = \varepsilon_{22}^0 = 0$ and, consequently, $\sigma_{33}^0 = (2\mu + \lambda)\varepsilon_{33}^0$. Then, in the case of a small transverse stretching (9.2), using the first formula of (6.3) and the last formula of (9.2), we shall have

$$\tilde{\varepsilon}_{33} = p_0 \varepsilon_{33}^0 / p_0^*(h_1^*) \tag{9.5}$$

Relation (9.5) shows that, at the initial stage of the transverse stretching of a laminated medium (corresponding to the segment (7.12)), the average strain $\tilde{\varepsilon}_{33}$ increases and, when $\tilde{\sigma}_{33} = p_0$, it reaches its maximum value

$$\tilde{\varepsilon}_{33}^{\max} = (2\mu + \lambda)^{-1} p_0 \tag{9.6}$$

According to the penultimate formula of (9.3), further stretching (corresponding to the segment (7.13)) leads to the following value of the average transverse strain

$$\tilde{\varepsilon}_{33} = p_0(\varepsilon_{33}^0 - H^{-1} h_0^*) / p_0^*(h_1^* - h_0^*) \tag{9.7}$$

Relation (9.7) shows that, when the stresses σ_{33}^0 are varied within the limits of the bounds of the double inequality (9.3) or, what is the same thing, when

$$(2\mu + \lambda)^{-1} p_0^*(h_1^*) \le \varepsilon_{33}^0 \le H^{-1} h_0^* \tag{9.8}$$

the average strain $\tilde{\varepsilon}_{33}$ varies within the limits from the value determined by formula (9.7) to zero.

We see that the decrease in the value of (9.7) when the stretching is increased within the limits of (9.8) is explained by the fact that the value of $\tilde{\varepsilon}_{33}$ determines the average strain in the elastic layers, while the value of ε_{33}^0 determines the deformation of the whole laminated medium between the layers of which fissures have appeared.

Finally, according to the penultimate formula of (9.4), when the adhesive bonds have been overcome, we obtain $\tilde{\varepsilon}_{33} = 0$ which, when account is taken of the last formula of (9.4), means that there is no transverse strain in the elastic layers, that is, the layers have simply separated from one another to a distance at which adhesive forces do not act.

10. Concluding remarks

In the case of rough layers, Coulomb's law[4] can be generalized by taking account of the prior elastic displacements which are observed experimentally when $\boldsymbol{\sigma}_\tau + f\sigma_{33} < 0$ and correspond to shear deformations of the microroughnesses (see Refs 26 and 27, for example).

Unlike Signorini condition (1.9), boundary condition (1.3) permits mutual penetration of the surfaces of the elastic layers but links its value with the magnitude of the compressive stresses. Actually, mutual penetration does not occur in the case of elastic contact of microroughnesses. Speaking more precisely, the local deformation of the microscopic unevenesses of rough surfaces is modelled as the result of a certain penetration of the nominal surfaces of the contacting bodies. When necessary, it is possible to get rid of the simplifying assumption concerning the zero thickness of the adhesive seam made in Section 2. In boundary condition (2.3), it is possible, together with a certain initial gap which is equal to the thickness of the adhesive seam, also to take account of the compliance of the adhesive seam under compression.

It has already been mentioned that Eq. (2.1) describes the irreversible process of the breakdown of the adhesive seam. Addition to the right-hand side of Eq. (2.1) of a positive additive term enables one to describe the process of the restoration of the adhesive bonds when $[u_3] = 0$ (see Ref. 13, for example).

The appearance of the parameter ε on the right-hand side of Eq. (5.9) is indicative of the "slowness" of the course of the adhesive seam breakdown process in a finely stratified packet. However, we emphasize that, in the quasistatic problem considered, it is only possible to separate out one complex (which includes the parameter γ_{de}) having the dimension of time. For a correct description of the rate of the breakdown process, it is thereby necessary to a introduce a certain parameter characterizing the rate of breakdown of the packet of layers.

The Dugdale model[28], proposed to describe the small plastic zone in the terminal region of a crack, is associated with the piecewise-constant approximation (7.3). We recall (in particular, see Ref. 29) that, in the Dugdale scheme for modelling the narrow plastic zone extending a real crack, it is assumed that stresses act in the terminal zones of a (mathematical) crack which are equal to the yield stress. Since, in the Leonov–Panasyuk[30] model, it is assumed that, in the end zone of a crack, its faces are attracted by surface forces to which stresses equal to the theoretical strength correspond. At the same time, as has been pointed out in Ref. 29, (p. 156), this piecewise-constant hypothesis is a version of the approximation of the graph of the interatomic interaction forces. Hence, the piecewise-constant approximation of Maugis (7.3) is closer to the Leonov–Panasyuk model than the Dugdale model.

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